THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 10 (April 22)

Boundedness Theorem. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I. Then f is bounded on I.*

Extreme Value Theorem. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I. Then f has an absolute maximum and an absolute minimum on I, that is, there exist $x_*, x^* \in I$ such that*

$$f(x_*) \le f(x) \le f(x^*) \qquad \text{for all } x \in I.$$

Intermediate Value Theorem. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I. If $f(a) < k < f(b)$ (or $f(b) < k < f(a)$), then there exists $c \in (a, b)$ such that $f(c) = k$.*

Example 1. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots.

Solution. Since p is continuous on $[0, 2]$ and $p(0) = -9 < 0 < 63 = p(2)$, it follows from the Intermediate Value Theorem that $p(c_1) = 0$ for some $c_1 \in (0, 2)$.

Since p is continuous on $[-8, 0]$ and $p(-8) = 503 > 0 > -9 = p(0)$, it follows from the Intermediate Value Theorem that $p(c_2) = 0$ for some $c_2 \in (-8, 0)$.

As $c_1 \neq c_2$, p has at least two real roots. ◀

Example 2. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists a point c in $[0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

Solution. Let $g(x) := f(x) - f(x + \frac{1}{2})$. Then g is a continuous function on $[0, 1]$ such that

$$g(0) = f(0) - f(\frac{1}{2}) = f(1) - f(\frac{1}{2}) = -g(\frac{1}{2}).$$

If $g(0) = 0$, then simply take $c = 0$. Otherwise, 0 is between $g(0)$ and $g(\frac{1}{2})$. Hence, by the Intermediate Value Theorem there exists $c \in (0, \frac{1}{2})$ such that $g(c) = 0$, that is

$$f(c) = f(c + \frac{1}{2}).$$

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Example 3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim\limits_{x\to-\infty} f = 0$ and $\lim\limits_{x\to\infty} f = 0$. Prove that f attains either a maximum or minimum on $\mathbb{R}$.

Solution. **Case 1:** If $f \equiv 0$, then f attains both a maximum and a minimum at any point.

Case 2: Suppose $f \not\equiv 0$. Then there is $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. WLOG, assume $f(x_0) > 0$. We will show that f attains a maximum on $\mathbb{R}$. Take $\varepsilon_0 = f(x_0)/2$. Since $\lim_{x\to-\infty} f = 0$ and $\lim_{x\to\infty} f = 0$, there is $K > 0$ such that

$$|f(x)| < \varepsilon_0 \quad \text{whenever } |x| > K.$$

Let $K' = \max\{K, |x_0|\}$. Since f is continuous on $[-K', K']$, it follows from the Extreme Value Theorem that there exist $x_*, x^* \in [-K', K']$ such that

$$f(x_*) \leq f(x) \leq f(x^*) \quad \text{for all } x \in [-K', K'].$$

Moreover, if $|x| > K'$, then

$$f(x) < \varepsilon_0 < f(x_0) \leq f(x^*).$$

Combining the inequalities, we have $f(x) \leq f(x^*)$ for all $x \in \mathbb{R}$. Hence f attains a maximum on $\mathbb{R}$. ◀

Classwork

1. Let $f\colon [0,1] \to [0,1]$ be a continuous function. Show that there exists some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

   Solution. Let $g(x) = f(x) - x$. Then g is a continuous function on $[0,1]$. Moreover $g(0) = f(0) - 0 \geq 0 - 0 = 0$ and $g(1) = f(1) - 1 \leq 1 - 1 = 0$.

   If $g(0) = 0$ (or $g(1) = 0$), then take $x_0 = 0$ (or $x_0 = 1$).

   If $g(1) < 0 < g(0)$, then the Intermediate Value Theorem implies that there is $x_0 \in (0,1)$ such that $g(x_0) = 0$.

   In either case, we have $f(x_0) = x_0$. ◀

2. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous and that $\lim_{x\to\infty} f = 0$. Prove that f is bounded.

   Solution. Take $\varepsilon_0 = 1$. As $\lim_{x\to\infty} f = 0$, there is $K > 0$ such that

   $$|f(x)| < \varepsilon_0 = 1 \quad \text{whenever } x > K.$$

   Since f is continuous on $[0, K]$, it follows from the Boundedness Theorem that there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, K]$. Now f is bounded on $[0,\infty)$ because

   $$|f(x)| \leq M + 1 \quad \text{for all } x \in [0,\infty).$$

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